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AN ALGORITHM FOR NORMALIZING HAMILTONIAN SYSTEMS IN THE PROBLEM OF THE ORBITAL STABILITY OF PERIODIC MOTIONS†

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A constructive procedure is proposed for constructing equations of perturbed motion convenient for investigating the orbital stability of periodic motion in an autonomous Hamiltonian system with two degrees of freedom. An algorithm for normalizing these equations is described, and formulae for evaluating the coefficients of the normal form are presented. The results are used to investigate the stability of motion in certain special cases of the regular Grioli precession of a heavy rigid body with one fixed point. © 2003 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a system with two degrees of freedom whose motion is described by equations

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i},\quad i=1,2 \tag{1.1}$$

where the Hamiltonian H is independent of the time t.

Let us assume that the system admits of a periodic motion

$$q_i=f_i(t),\quad p_i=g_i(t),\quad i=1,2 \tag{1.2}$$

Without loss of generality, we may assume that the period is 2π. We shall also assume that the Hamiltonian is analytic in the neighbourhood of the trajectory corresponding to the periodic motion.

To investigate the motions of the system near the periodic motion, it is convenient to introduce canonically conjugate variables ξ_i, η_i (i = 1, 2), in such a way that the unperturbed motion (1.2) may be written as

$$\xi_1(t)=t+\xi_1(0),\quad \eta_1=\xi_2=\eta_2=0 \tag{1.3}$$

and the Hamiltonian is 2π-periodic in ξ_1.

The existence of such variables has long been known [1, 2], but their actual construction in specific problems of dynamics may prove to be an extremely complicated task.

This problem is comparatively easy to solve (see, e.g. [3–5]) if the Hamiltonian of system (1.1) can be written in the form

$$H=H^{(1)}(q_1,p_1)+H^{(2)}(q_1,q_2,p_1,p_2) \tag{1.4}$$

where the function $H^{(2)}$ is expressible in the form of a series in powers of q_2 and p_2 beginning with terms of degree at least two. System (1.1) with Hamiltonian (1.4) has a family of solutions for which $q_2=p_2=0$, and the variables p_1 and q_1 are described by equations corresponding to a system with one degree of freedom having the Hamiltonian $H^{(1)}$.

In domains of the phase space q_1, p_1 where the motions of the system have the periodicity property we introduce action-angle variables I and w. The periodic motion is expressed in variables I and w as

$$w=\omega(I_0)t+w_0,\quad I=I_0$$

where $\omega(I)=dh^{(1)}/dI$, $h^{(1)}$ being the Hamiltonian $H^{(1)}$ expressed in terms of I and w. For a 2π-periodic motion, $\omega(I_0)=1$. If we put

$$\xi_1=w,\quad \eta_1=I-I_0,\quad \xi_2=q_2,\quad \eta_2=p_2$$

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the periodic motion of the original system with Hamiltonian (1.4) is written in the form (1.3), and Hamiltonian (1.4) expressed in terms of variables ξ_i and η_i will be 2π-periodic with respect to ξ_i and analytic in ξ_2, η_1 and η_2.

The aim of this paper is to derive a method for constructing the variables ξ_i and η_i for the general case, when the Hamiltonian of the system does not possess the structure (1.4), and also to develop a constructive algorithm for normalizing the Hamiltonian of the perturbed motion. As an example, we shall investigate the stability of the regular Grioli precession of a rigid body with a fixed point in a uniform gravitational field, in a few special cases.

2. DERIVATION OF THE HAMILTONIAN OF PERTURBED MOTION

The variables ξ_i and η_i will be introduced using a canonical univalent transformation $q_i, p_i \to \xi_i, \eta_i$. In so doing we shall confine our attention to the class of transformations that are linear in the variables ξ_2, η_1 and η_2 characterizing the deviation of the perturbed trajectories of system (1.1) from the trajectories of the unperturbed periodic motion (1.2). We set

$$\begin{aligned} q_i &= f_i(\xi_1)+a_{i1}(\xi_1)\xi_2+a_{i2}(\xi_1)\eta_1+a_{i3}(\xi_1)\eta_2 \\ p_i &= g_i(\xi_1)+b_{i1}(\xi_1)\xi_2+b_{i2}(\xi_1)\eta_1+b_{i3}(\xi_1)\eta_2; \quad i=1,2 \end{aligned} \tag{2.1}$$

where f_i and g_i are the functions of (1.2), and a_{ij} and b_{ij} ($i = 1, 2; j = 1, 2, 3$) are unknown 2π-periodic functions of ξ_1, to be chosen in such a way that transformation (2.1) is canonical univalent.

Let $S = S(\xi_1, \xi_2, p_1, p_2)$ be the generating function of transformation (2.1). The relation between the old variables and the new is given by the equalities

$$q_i = \frac{\partial S}{\partial p_i}, \quad \eta_i = \frac{\partial S}{\partial \xi_i}, \quad i=1,2 \tag{2.2}$$

Since the substitution (2.1) is linear with respect to ξ_2, η_1 and η_2, the generating function must be a second-degree function in these variables, in which the coefficients of the second-degree terms are constants. Let us write the function S as

$$S = c_1\xi_2^2+c_2\xi_2p_1+c_3\xi_2p_2+c_4p_1^2+c_5p_1p_2+c_6p_2^2+s_1\xi_2+s_2p_1+s_3p_2+\int s_0 d\xi_1$$

where $c_1, c_2, \ldots, c_6$ are constant coefficients and s_0, s_1, s_2, s_3 are 2π-periodic functions of ξ_1.

Having solved Eqs (2.2) for the variables q_1, q_2, p_1, p_2 and substituted the resulting expressions into the left-hand sides of (2.1), we find that

$$\begin{aligned} s_0 &= (c_4g_1^2+c_5g_1g_2+c_6g_2^2)'-f_1'g_1-f_2'g_2, \quad s_1 = -c_2g_1-c_3g_2 \\ s_2 &= f_1-2c_4g_1-c_5g_2, \quad s_3 = f_2-c_5g_1-2c_6g_2 \end{aligned} \tag{2.3}$$

and the following expressions are found for the coefficients $a_{ij}(\xi_1)$, $b_{ij}(\xi_1)$ of transformation (2.1)

$$\begin{aligned} a_{i1} &= (-1)^{i+1}\Delta^{-1}[e_1f_i'-e_{3+i}f_{3-i}'-2(e_1c_5-e_4c_6+c_1c_5^2-c_3^2c_4)g_{3-i}'] \\ a_{i2} &= -(-1)^{i+1}\Delta^{-1}e_{4-i}, \quad a_{i3} = (-1)^{i+1}\Delta^{-1}(c_5f_i'-2c_{2+2i}f_{3-i}'-e_6g_{3-i}') \\ b_{i1} &= (-1)^{i+1}\Delta^{-1}(2c_1f_{3-i}'+e_1g_i'+e_{6-i}g_{3-i}') \\ b_{i2} &= (-1)^{i+1}\Delta^{-1}c_{4-i}, \quad b_{i3} = (-1)^{i+1}\Delta^{-1}(-f_{3-i}'+c_5g_i'+2c_{8-2i}g_{3-i}') \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &e_1 = c_2c_3-2c_1c_5, \quad e_2 = c_3c_5-2c_2c_6, \quad e_3 = c_2c_5-2c_3c_4 \\ &e_4 = c_2^2-4c_1c_4, \quad e_5 = c_3^2-4c_1c_6, \quad e_6 = c_5^2-4c_4c_6 \\ &\Delta = c_3f_1'-c_2f_2'+e_3g_1'-e_2g_2' \end{aligned} \tag{2.5}$$

The prime denotes differentiation with respect to ξ_1.

The change of variables (2.1) that we have obtained contains six arbitrary constant parameters c_1, c_2, ..., c_6. The choice of these parameters when solving specific problems is governed by the condition that the quantity Δ, defined by the last formula of (2.5), should not vanish for $0 \leqslant \xi_1 \leqslant 2\pi$.

Example 1. Suppose one of the phase coordinates in periodic motion (1.2), say q_1, is a monotone function of time ($\dot{f}_1(t) \neq 0$). In that case the transformation $q_i, p_i \to \xi_i, \eta_i$ may be obtained by setting $c_3 = 1$ in formulae (2.4) and equating the other five parameters c_i to zero. Then the change of variables (2.1) will be

$$q_1 = f_1(\xi_1), \quad q_2 = f_2(\xi_1) + \xi_2$$
$$p_1 = g_1(\xi_1) + (f_1')^{-1}(\eta_1 + g_2'\xi_2 - f_2'\eta_2), \quad p_2 = g_2(\xi_1) + \eta_2 \tag{2.6}$$

Remark. It is not hard to verify that transformation (2.1) can be expressed as a composition of two canonical univalent changes of variables. In the first transformation $q_i, p_i \to Q_i, P_i$, we introduce a new coordinate

$$Q_1 = c_3 q_1 - c_2 q_2 + e_3 p_1 - e_2 p_2$$

The second transformation $Q_i, P_i \to \xi_i, \eta_i$ is analogous to transformation (2.6).

Example 2. Suppose for the periodic motion (1.2) $\dot{f}_1(t) + \dot{g}_2(t) \neq 0$. Applying the transformation

$$Q_1 = q_1 + p_2, \quad Q_2 = q_2 + p_1, \quad P_1 = p_1, \quad P_2 = p_2$$

followed by a transformation similar to (2.6), we obtain the following transformation (2.1)

$$q_1 = f_1(\xi_1) - \eta_2, \quad q_2 = f_2(\xi_1) - (f_1' + g_2')^{-1}[\eta_1 - f_1'\xi_2 - (f_2' + g_1')\eta_2]$$
$$p_1 = g_1(\xi_1) + (f_1' + g_2')^{-1}[\eta_1 + g_2'\xi_2 - (f_2' + g_1')\eta_2], \quad p_2 = g_2(\xi_1) + \eta_2 \tag{2.7}$$

In terms of variables ξ_i and η_i, the unperturbed periodic motion (1.2) is written in the form of Eqs (1.3).

To obtain the Hamiltonian of the perturbed motion $\Gamma = \Gamma(\xi_1, \xi_2, \eta_1, \eta_2)$ from the Hamiltonian $H(q_1, q_2, p_1, p_2)$ of the original system (1.1), the variables of q_i and p_i must be replaced by their expressions in terms of ξ_i and η_i as in formulae (2.1).

The function Γ may be expanded in a convergent series in powers of η_1, ξ_2 and η_2

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4 + \ldots + \Gamma_k + \ldots \tag{2.8}$$

where we have omitted an unimportant additive constant, equal to the value of the Hamiltonian for the unperturbed motion (1.2), and Γ_k is a form of degree k in $|\eta_1|^{1/2}$, ξ_2 and η_2, with

$$\Gamma_2 = \eta_1 + \varphi_2(\xi_2, \eta_2, \xi_1), \quad \Gamma_3 = \psi_1(\xi_2, \eta_2, \xi_1)\eta_1 + \varphi_3(\xi_2, \eta_2, \xi_1)$$
$$\Gamma_4 = \chi(\xi_1)\eta_1^2 + \psi_2(\xi_2, \eta_2, \xi_1)\eta_1 + \varphi_4(\xi_2, \eta_2, \xi_1) \tag{2.9}$$

where $\chi(\xi_1)$ is a 2π-periodic function of ξ_1, φ_m and ψ_m are forms of degree m in ξ_2 and η_2 whose coefficients are 2π-periodic functions of ξ_1.

The Hamiltonian (2.8) is convenient for investigating trajectories of system (1.1) close to trajectories of the unperturbed periodic motion (1.2) [6]. In particular, the problem of the orbital stability of (1.2) is equivalent to the problem of the stability of the system with Hamiltonian (2.8) relative to perturbations of η_1, ξ_2 and η_2.

3. NORMALIZATION OF THE HAMILTONIAN. FORMULATION OF THE CONDITIONS FOR ORBITAL STABILITY AND INSTABILITY

Corresponding to the linearized equations of perturbed motion we have the Hamiltonian Γ_2, defined by the first expression of (2.9). Two multipliers of these equations equal unity; the other two are the roots of the equation

$$\rho^2 - 2a\rho + 1 = 0 \tag{3.1}$$

where $2a = x_{11}(2\pi) + x_{22}(2\pi)$, and $x_{ii}(2\pi)$ are elements, evaluated at $\xi_1 = 2\pi$, of the matrix $\mathbf{X}(\xi_1)$ of fundamental solutions ($\mathbf{X}(0, 0) = \mathbf{E}$, where $\mathbf{E}$ is the 2×2 identity matrix) of a linear system whose coefficients are 2π-periodic in the independent variable ξ_1

$$\frac{d\xi_2}{d\xi_1} = \frac{\partial\varphi_2}{\partial\eta_2}, \quad \frac{d\eta_2}{d\xi_1} = -\frac{\partial\varphi_2}{\partial\xi_2} \tag{3.2}$$

where φ_2 is the part of Γ_2 that is quadratic in ξ_2 and η_2.

Normalization of the Hamiltonian (2.8) is achieved differently, depending on the value of a.

If $|a| > 1$, Eq. (3.1) has one multiplier whose absolute value exceeds unity. Then the unperturbed periodic motion is orbitally unstable by Lyapunov's theorem on stability in the first approximation [7].

If $|a| = 1$, Eq. (3.1) has multiple real roots $\rho_1 = \rho_2 = 1$ (if $a = 1$) or $\rho_1 = \rho_2 = -1$ (if $a = -1$). In the general position, the matrix $\mathbf{X}(2\pi)$ cannot be diagonalized and, consequently, the unperturbed periodic motion is orbitally unstable in the first approximation. A rigorous solution of the stability problem in this case requires and examination of the non-linear equations of the perturbed motion. In a previous paper [8] a constructive algorithm for normalizing the Hamiltonian of perturbed motion (2.8) in the case $|a| = 1$ was developed, and also conditions for orbital stability and instability were obtained. The normalized Hamiltonian has the form

$$H = r_1 + \tfrac{1}{2}\delta y_2^2 + k_{30}x_2^3 + k_{10}x_2 r_1 + k_{40}x_2^4 + k_{20}x_2^2 r_1 + k_{00}r_1^2 + O_5 \tag{3.3}$$

where k_{ij} are constants, and O_5, is a series beginning with terms of degree at least five in $|r_1|^{1/2}, x_2$ and y_2, whose coefficients have period 2π (if $a - 1$) or 4π (if $a = -1$) relative to the coordinate w_1 corresponding to momentum r_1. The number δ in (3.3) is 1 or -1, its actual value being determined when normalizing the linear system (3.2). Formulae for the coefficients of the normal form (3.3) were presented in [8].

Theorem 1. If the coefficient k_{30} of the normal form (3.3) does not vanish, or if $k_{30} = 0$ but $\delta k_{40} < 0$, then the periodic motion is orbitally unstable.

Theorem 2. If the coefficient k_{30} of the normal form (3.3) vanishes, but at the same time $\delta k_{40} > 0$, then the periodic motion is orbitally stable.

Now let $|a| < 1$. In that case the roots of Eq. (3.1) are distinct and of absolute value unity: $\rho_1 = e^{i2\pi\lambda}$, $\rho_2 = e^{-i2\pi\lambda}$, where λ is a root of the equation

$$\cos 2\pi\lambda = a \tag{3.4}$$

In specific problems, the non-uniqueness in the value of λ in this equation may often be eliminated by considering limiting cases in which the quadratic form φ_2 in (2.9) has constant coefficients. In such limiting cases, $|\lambda|$ will be the frequency of small oscillations and is easily evaluated. Then, taking into account that the characteristic exponents are continuous functions of the parameters of the problem, one can uniquely determine λ from Eq. (3.4).

Normalization of the quadratic part of Hamiltonian (2.8). Suppose the characteristic exponents $\pm i\lambda_*$, of system (3.2) have been found. Set $\varkappa = x_{12}(2\pi)\sin(2\pi\lambda_*)$, $\sigma = \operatorname{sign}\varkappa$, $\lambda = \sigma\lambda_*$, and make the change of variables $\xi_1, \xi_2, \eta_1, \eta_2, \to u_1, u_2, v_1, v_2$ defined by the formulae

$$\xi_1 = u_1, \quad \eta_1 = v_1 + 1/2\lambda(u_2^2 + v_2^2) - \varphi_2(n_{11}u_2 + n_{12}v_2,\ n_{21}u_2 + n_{22}v_2,\ \xi_1) \tag{3.5}$$

$$\xi_2 = n_{11}u_2 + n_{12}v_2, \quad \eta_2 = n_{21}u_2 + n_{22}v_2$$

The matrix $\mathbf{N}(u_1)$ of the coefficients n_{ij} of this transformation is defined by the following equalities

$$\mathbf{N} = \sigma_1\mathbf{X}(u_1)\mathbf{P}\mathbf{Q}(u_1)$$

$$\mathbf{P} = \begin{Vmatrix} c & 0 \\ d & c^{-1} \end{Vmatrix}, \quad \mathbf{Q} = \begin{Vmatrix} \cos\lambda u_1 & -\sin\lambda u_1 \\ \sin\lambda u_1 & \cos\lambda u_1 \end{Vmatrix}$$

$$\sigma_1 = \operatorname{sign}(\sin 2\pi\lambda_*), \quad c = x_{12}(2\pi)|\varkappa|^{-1/2}, \quad d = (\cos 2\pi\lambda - x_{11}(2\pi))|\varkappa|^{-1/2}$$

Transformation (3.5) is canonical, univalent and 2π-periodic in u_1. After making replacement (3.5) one can write Hamiltonian (2.8) in the form

$$F = F_2 + F_3 + F_4 + \ldots + F_k + \ldots \tag{3.6}$$
$$F_2 = v_1 + \tfrac{1}{2}\lambda(u_2^2 + v_2^2), \quad F_3 = f_1(u_2, v_2, u_1)v_1 + f_3(u_2, v_2, u_1)$$
$$F_4 = \chi(u_1)v_1^2 + f_2(u_2, v_2, u_1)v_1 + f_4(u_2, v_2, u_1)$$

where f_k is a form of degree k in u_2 and v_2 whose coefficients are 2π-periodic functions of u_1

$$f_k = \sum_{\nu+\mu=k} f_{\nu\mu}(u_1)u_2^\nu v_2^\mu$$

Derivation of the normal form up to terms of degree four inclusive. The terms of the third and fourth degree in Hamiltonian (3.6) will be normalized by the Deprit–Hori method [9, 10]. A canonical normalizing transformation $u_1, v_1, u_2, v_2 \to w_1, r_1, q_2, p_2$ can be obtained close to the identical transformation

$$u_1 = w_1 + \ldots, \quad v_1 = r_1 + \ldots, \quad u_2 = q_2 + \ldots, \quad v_2 = p_2 + \ldots$$

where the dots stand for convergent series in powers of r_1, q_2 and p_2 whose coefficients are 2π-periodic functions of w_1.

The normal form of the Hamiltonian will be different, depending on whether there is resonance of order 3 or 4 (i.e. whether one of the quantities 3λ or 4λ is an integer k) or whether there is no such resonance.

Without dwelling on the rather cumbersome calculations, we shall present the final form of the formulae necessary to compute the normal form. We introduce the notation

$$\delta_{10}(u_1) = 2f_{10}, \quad \gamma_{10}(u_1) = 2f_{01}, \quad \delta_{30}(u_1) = f_{03} - f_{21}$$
$$\gamma_{30}(u_1) = f_{30} - f_{12}, \quad \delta_{21}(u_1) = -(3f_{03} + f_{21}), \quad \gamma_{21}(u_1) = 3f_{30} + f_{12}$$
$$l_{\nu\mu} = \frac{1}{4}\int_0^{u_1} [\delta_{\nu\mu}(t)\cos(\nu-\mu)\lambda t - \gamma_{\nu\mu}(t)\sin(\nu-\mu)\lambda t]dt$$
$$m_{\nu\mu} = \frac{1}{4}\int_0^{u_1} [\delta_{\nu\mu}(t)\sin(\nu-\mu)\lambda t + \gamma_{\nu\mu}(t)\cos(\nu-\mu)\lambda t]dt$$
$$r_{\nu\mu} = 2l_{\nu\mu}(u_1) + m_{\nu\mu}(2\pi)\operatorname{ctg}[(\nu-\mu)\pi\lambda] - l_{\nu\mu}(2\pi)$$
$$s_{\nu\mu} = 2m_{\nu\mu}(u_1) - l_{\nu\mu}(2\pi)\operatorname{ctg}[(\nu-\mu)\pi\lambda] - m_{\nu\mu}(2\pi)$$
$$u_{\nu\mu} = \tfrac{1}{2}[r_{\nu\mu}\cos(\nu-\mu)\lambda u_1 + s_{\nu\mu}\sin(\nu-\mu)\lambda u_1]$$
$$v_{\nu\mu} = -\tfrac{1}{2}[r_{\nu\mu}\sin(\nu-\mu)\lambda u_1 - s_{\nu\mu}\cos(\nu-\mu)\lambda u_1]$$
$$d_{20} = f_{10}v_{10} - f_{01}u_{10}, \quad d_{11} = -2(f_{10}u_{21} + f_{01}v_{21})$$
$$d_{40} = f_{10} - 3\lambda v_{10} - 3u_{21}, \quad d_{04} = f_{01} + 3\lambda u_{10} - 3v_{21}$$

If there are no resonances of order 3 and 4, the normalized Hamiltonian is

$$H = r_1 + \lambda r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + O_5 \tag{3.7}$$

In (3.7) (and later) $q_2 = \sqrt{2r_2}\sin w_2, p_2 = \sqrt{2r_2}\cos w_2$, while O_5 is the sum of all terms of degree at least five in $|r|^{1/2}$, q_2 and p_2. The constant coefficients c_{ij} are evaluated by the formulae

$$c_{20} = \langle \chi + d_{20} \rangle, \quad c_{11} = \langle f_{20} + f_{02} - f_{10}^2 - f_{01}^2 + \delta_{21}u_{10} + \gamma_{21}v_{10} + d_{11} - 2\lambda d_{20} \rangle$$
$$c_{02} = \tfrac{1}{2}\langle 3f_{40} + f_{22} + 3f_{04} - \lambda d_{11} - 9\gamma_{30}u_{30} + 9\delta_{30}v_{30} +$$
$$+ \delta_{21}(f_{01} - \lambda u_{10} + 3v_{21}) - \gamma_{21}(f_{10} + \lambda v_{10} + 3u_{21})\rangle$$

The symbol $\langle g\rangle$ denotes the average of a 2π-periodic function $g(u_1)$ over a period.
Let

$$D = c_{20}\lambda^2 - c_{11}\lambda + c_{02} \tag{3.8}$$

Theorem 3 (Arnol'd–Moser) [11]. If $D \neq 0$, the periodic motion is orbitally stable.

For resonance of order there, $3\lambda = k$, we have the following normal form for the Hamiltonian of perturbed motion

$$H = r_1 + \lambda r_2 + r_2\sqrt{r_2}[\alpha_{30}\sin(3w_2 - kw_1) + \beta_{30}\cos(3w_2 - kw_1)] + O_4 \tag{3.9}$$

$$\alpha_{30} = -\frac{\sqrt{2}}{2}\langle\delta_{30}\sin ku_1 + \gamma_{30}\cos ku_1\rangle, \quad \beta_{30} = \frac{\sqrt{2}}{2}\langle\delta_{30}\cos ku_1 - \gamma_{30}\sin ku_1\rangle$$

Theorem 4 [10]. If at least one of the coefficients α_{30} or β_{30} of the Hamiltonian (3.9) does not vanish, the periodic motion is orbitally unstable.

For resonance of order four, $4\lambda = k$, the normal form will be

$$\begin{aligned} H &= r_1 + \lambda r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + \\ &+ r_2^2[\alpha_{40}\sin(4w_2 - kw_1) + \beta_{40}\cos(4w_2 - kw_1)] + O_5 \end{aligned} \tag{3.10}$$

The quantities c_{ij} in (3.10) are evaluated like those in the normal form (3.7), but

$$\alpha_{40} = -\tfrac{1}{2}\langle\sigma_{40}\sin ku_1 - \chi_{40}\cos ku_1\rangle, \quad \beta_{40} = \tfrac{1}{2}\langle\sigma_{40}\cos ku_1 + \chi_{40}\sin ku_1\rangle$$

$$\sigma_{40} = f_{40} - f_{22} + f_{04} - \gamma_{30}d_{40} - \delta_{30}d_{04} + 6\lambda(f_{10}u_{30} - f_{01}v_{30}) - 3(\delta_{21}v_{30} + \gamma_{21}u_{30})$$

$$\chi_{40} = f_{13} - f_{31} - \delta_{30}\delta_{40} + \gamma_{30}d_{04} - 6\lambda(f_{10}v_{30} + f_{01}u_{30}) - 3(\delta_{21}u_{30} - \gamma_{21}v_{30})$$

Theorem 5 [10]. If $|D| > \sqrt{\alpha_{40}^2 + \beta_{40}^2}$, the periodic motion is orbitally stable. If the inverse inequality holds, the periodic motion is orbitally unstable.

4. ANALYSIS OF THE STABILITY GRIOLI PRECESSION IN TWO SPECIAL CASES

Let us consider the motion of a rigid body with a fixed point O is a uniform gravitational field. The weight of the body is mg and the distance from its centre of gravity to the fixed point is l. Suppose the fixed point has been chosen so that

$$\tilde{x}_0\sqrt{B-C} = \tilde{z}_0\sqrt{A-B}, \quad \tilde{y}_0 = 0, \quad A > B > C \tag{4.1}$$

where $\tilde{x}_0, \tilde{y}_0, \tilde{z}_0$ are the coordinates of the centre of gravity in a system of coordinates $O\tilde{x}_0\tilde{y}_0\tilde{z}_0$ whose axes are the principal axes of inertia of the body for the fixed point, A, B and C being the corresponding moments of inertia. Conditions (4.1) mean that the body possesses no dynamic symmetry, while the centre of gravity lies on the perpendicular to a circular section of the inertia ellipsoid constructed from the fixed point.

Grioli showed [12] (see also [13–16]) that, if condition (4.1) is satisfied, the body may precess regularly about an axis other than the vertical. The angle χ between the axis of precession and the upward vertical is uniquely defined by the values of the principal moments of inertia

$$\chi = \operatorname{arctg} b, \quad b = \frac{\sqrt{(1-\theta_b)(\theta_b - \theta_c)}}{1 - \theta_b + \theta_c}$$

where $\theta_b = B/A$ and $\theta_c = C/A$ are non-dimensional parameters. In the θ_b, θ_c plane, the domain of their admissible values ($0 < \theta_c < \theta_b < 1$; $\theta_b + \theta_c > 1$) is a right-angled triangle with vertices ($^1/_2$, $^1/_2$), (1, 1) and (0, 0).

In Grioli precession, the centre of gravity of the body lies on its spinning axis, and the angle between the axes of the fixed and non-fixed avoids is a right angle. The angular velocities of precession and spinning are the same, both equalling a number n that depends solely on the mass geometry of the body

$$n^2 = \frac{mgl}{\sqrt{(A-B)(B-C)+(A-B+c)^2}}$$

The motion of the body corresponding to regular Grioli precession is periodic: in a time $2\pi/n$ the body returns to its initial orientation in absolute space, and at the same time the angular velocity vector takes its initial value.

The algorithms of Sections 1–3 have been used to solve the problem of the stability of Grioli precession. The results will be presented below for two special cases. Incidentally, some questions relating to the stability of Grioli precession have been considered before [17–19].

We introduce a trihedron $Oxyz$ rigidly attached to the body, obtained from the trihedron $O\tilde{x}_0\tilde{y}_0\tilde{z}_0$ by counterclockwise rotation about the $O\tilde{y}_0$ axis through the angle $\alpha = \operatorname{arctg}(\tilde{x}_0/\tilde{z}_0)$. The Oz axis passes through the body's centre of gravity. The orientation of the trihedron $Oxyz$ is defined by the Euler angles ψ, θ and φ. To obtain the Hamiltonian, we put

$$\varphi = q_1, \quad \theta = q_2, \quad \psi = q_3, \quad p_\varphi = Anp_1, \quad p_\theta = Anp_2, \quad p_\psi = Anp_3$$

taking the dimensionless quantity $\tau = n(t + t_0)$, where t_0 is an arbitrary constant, as the independent variable. Without writing down the expressions for the Hamiltonian, it suffices to note that q_3 is a cyclic coordinate, and that Grioli precession is represented by the following solution of the reduced autonomous Hamiltonian system with two degrees of freedom

$$\begin{aligned}
&q_1 = f_1(\tau) = -\frac{\pi}{2} + \tau - \operatorname{arctg}(b\sin\tau), \quad q_2 = f_2(\tau) = \arccos\frac{b\cos\tau}{\sqrt{b^2+1}} \\
&p_1 = g_1(\tau) = (1-\theta_b+\theta_c)(1-b\cos\tau) \\
&p_2 = g_2(\tau) = \frac{b\sin\tau}{\sqrt{1+b^2\sin^2\tau}}[1+\theta_c-(1-\theta_b+\theta_c)b\cos\tau]
\end{aligned} \tag{4.2}$$

Stability in the case when the axis of precession is inclined to the vertical at an angle of $\pi/4$. The central feature of this case is that in the θ_b, θ_c plane the curve $\chi = \pi/4$ divides the domain of admissible parameter values into two subdomains, in one of which ($\chi < \pi/4$) the angle of spin φ in the unperturbed motion (4.2) increases monotonically ($\dot{f}_1 > 0$), while in the other ($\chi > \pi/4$) the derivative $\dot{f}_1$ may vanish, and the angle φ varies non-monotonically.

On the curve $\chi = \pi/4$, the derivative $\dot{f}_1$ vanishes at $\tau = 0$ and $\tau = 2\pi$. However, the function $\dot{f}_1 + \dot{g}_2$ increases monotonically on the curve $\chi = \pi/4$ (calculations show that on this curve $\dot{f}_1 + \dot{g}_2 > 0.5$ for any τ). Therefore, by Example 2 in Section 1, the variables ξ_i and η_i may be introduced using canonical transformation (2.7).

The part of the curve $\chi = \pi/4$ lying in the domain of admissible parameter values may be defined by a single-valued function $\theta_c = \theta_c(\theta_b)$, with $5/6 < \theta_b < 1$. The stability is investigated for values of θ_b in the range (5.6, 0.999). To describe the results of the computations, we mark out nine points $Q^{(i)}$ in that range, numbering them in increasing order of the corresponding values $\theta_b^{(i)}$ of the parameter θ_b

$$\begin{aligned}
&\theta_b^{(1)} = 0.86442, \quad \theta_b^{(2)} = 0.87402, \quad \theta_b^{(3)} = 0.89662, \quad \theta_b^{(4)} = 0.90810, \quad \theta_b^{(5)} = 0.93519 \\
&\theta_b^{(6)} = 0.94849, \quad \theta_b^{(7)} = 0.97896, \quad \theta_b^{(8)} = 0.97936, \quad \theta_b^{(9)} = 0.99091
\end{aligned}$$

All the points in the range (5/6, 0.999) except for the six points $Q^{(j)}$ (j = 2, 3, 5, 6, 7, 9) are non-resonant: at these points the number $k\lambda$ is not an integer for k = 1, 2, 3, 4.

In the range ($\theta_b^{(5)}$, $\theta_b^{(6)}$), the coefficient a in Eq. (3.1) satisfies the inequality $a < -1$. At the endpoints of this range, $2\lambda = 3$, and the coefficients of the normal form (3.3) are $\delta = 1$, $k_{30} = 0$, $k_{40} = -0.00042$ at the point $Q^{(5)}$, $\delta = -1$, and $k_{30} = 0$, $k_{40} = -0.00037$ at the point $Q^{(6)}$.

At the points $Q^{(3)}$ and $Q^{(7)}$ there are third-order resonances $3\lambda = 5$ and $3\lambda = 4$, respectively. The coefficient β_{30} in the normal form (3.9) vanishes, $\alpha_{30} = -0.12572$ at the point $Q^{(3)}$ and $\alpha_{30} = -0.04245$ at the point $Q^{(7)}$.

At the point $Q^{(2)}$ and $Q^{(9)}$ there are fourth-order resonances $4\lambda = 7$ and $4\lambda = 5$, respectively. The coefficient α_{40} in the normal form (3.10) at these points vanishes, while β_{40} and the quantity D defined by (3.8) are equal respectively to –0.47388 and –0.14157 at the point $Q^{(2)}$, –0.07496 and –0.77175 at the point $Q^{(9)}$.

In the range (5/6, 0.999) the number D vanishes at three points: $Q^{(1)}$, $Q^{(4)}$ and $Q^{(8)}$.

Based on the material of Section 3 and the results of computations just presented, we reach the following conclusions concerning the orbital stability of Grioli precession for values of the parameters θ_b and θ_c on the curve $\chi = \pi/4$. At the points $Q^{(2)}$, $Q^{(3)}$, $Q^{(5)}$ and $Q^{(7)}$ and in the range $(\theta_b^{(5)}, \theta_b^{(6)})$, the motion is orbitally unstable; at the points $Q^{(1)}$, $Q^{(4)}$ and $Q^{(8)}$, the question of stability remains open, at all other points in the range (5/6, 0.999) Grioli precession is orbitally stable.

The limiting case $\theta_b + \theta_c = 1$. The part of the straight line $\theta_b + \theta_c = 1$ defined by $0.5 \leqslant \theta_b \leqslant 1$ is part of the boundary of the domain of admissible values of the parameters θ_b and θ_c. Investigation of the limiting case $\theta_b + \theta_c = 1$ may prove useful in analysing the motion of bodies which differ only slightly from a plate lying in the principle plane of inertia $O\bar{y}_0\bar{z}_0$ of the body.

The investigation was carried out for values of θ_b in the closed range [0.5, 0.99]. In this range, the function $\dot{f}_1 + \dot{g}_2$ is positive for all τ (computations showed that $\dot{f}_1 + \dot{g}_2 > 0.147$). The variables ξ_i and η_i, as in the case when $\chi = \pi/4$, may be introduced by using the change of variables (2.7).

The results of the numerical analysis are as follows. In the range [0.5, 0.99], we mark out three points $R^{(i)}$, to which the following values of the parameter correspond: θ_b: $\theta_b^{(1)} = 0.74957$, $\theta_b^{(2)} = 0.75652$, $\theta_b^{(3)} = 0.83902$. All points of the range under consideration except $R^{(1)}$ and $R^{(2)}$ are non-resonant.

In the range $(\theta_b^{(1)}, \theta_b^{(2)})$, the quantity a in Eq. (3.1) is greater than unity. At the endpoints $R^{(1)}$ and $R^{(2)}$ of that range, $\lambda = 2$, and in the normal form (3.3) we have the following coefficients: $\delta = 1$, $k_{30} = 0$, $k_{40} = -0.00018$ at the point $R^{(1)}$ and $k_{30} = 0.00008$ at the point $R^{(2)}$. The quantity D defined by (3.8) vanishes at the point $R^{(3)}$ but is non-zero at all other points in the range [0.5, 0.99].

Based on the material of Section 3, we conclude that in the limiting case $\theta_b + \theta_c = 1$, for values of θ_b in the range [0.5, 0.99], Grioli precession is orbitally unstable for $\theta_b^{(1)} \leqslant \theta_b \leqslant \theta_b^{(2)}$; for $\theta_b = \theta_b^{(3)}$ the question of stability remains open; for other values of θ_b, regular Grioli precession is orbitally stable.

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